A Dynamic Analysis for Elastic Structures Interacting with Rotating Machinery

Teh H. Lee*
Gulf General Atomic, San Diego, Calif.

A coupled modal method has been developed for analyzing the problem of dynamic interaction between flexible rotary machines and their elastic supporting structures. A special feature of the method considered here is that the structure modes are derived from a dynamic model which contains the rotary machines as rigid mass lumps; therefore, the customary component mode synthesis methods dealing with the assemblage of distinct components are not directly applicable. The present approach combines these types of structure modes directly with the free-free flexible modes of the rotary machines. An example problem is solved for the case under harmonic excitation produced by some interior source in the rotary machine, and the amplitude amplifications are determined for the supporting member loads. Interaction effects, including the influence of gyroscopic coupling, are investigated and discussed.

Nomenclature

A = rigid-body transformation matrix
B = rotary machine modal rotation matrix
f^G = gyroscopic moment vector; f_2^G and f_3^G are its components
$f(t)$ = vector of time-dependent point forces
$\bar{G}$ = system generalized gyroscopic matrix
I_1^F, I_2^F = fan polar moment of inertia and principal moment of inertia about diametrical axis
J = fan inertia matrix
$\bar{K}$ = system generalized stiffness matrix
k = structure stiffness matrix associated with u coordinate system
k^R = generalized stiffness matrix of rotary machine
k^{SS}, k^{SA}, k^{AS} = constituent submatrices of k associated with the structure part excluding the pylon
$k^{AA}, k^{AI}, k^{IA}, k^{II}$ = constituent submatrices for k for the pylon
m = mass matrix in physical coordinates
$\bar{M}$ = system generalized inertia matrix
Q = generalized force vector
q = system generalized modal coordinate vector
q^S = generalized modal coordinate vector associated with the structure modes
q^R = generalized modal coordinate vector associated with the rotary machine modes
T^1, T^2, T^3 = transformation matrices
T = kinetic energy
U = strain energy
u^A = vector of displacements of joints in the structure adjacent to u^I
u^I = vector of displacements of joints at the structure-rotor interface
u^R = rigid-body displacement vector for the mass center of the rotary machine
u^S = vector of displacements of joints in the structure excluding u^I and u^A
β = vector of rotation at fan station contributed from the elastic deformation of rotary machine
$\delta(\ )$ = virtual change in quantity ( )
δW = virtual work
η = vector of absolute displacements for points in the rotary machine
ξ = vector of absolute displacements for all mass points in the structure excluding the rigid mass of rotary machine
Φ = modal matrix of rotary machine
ϕ = modal matrix of structure
ψ = vector of rigid-body rotation of rotary machine; ψ_2 and ψ_3 are its components
Ω_r = rth natural frequency of rotary machine modes
ω_i = ith natural frequency of structure modes
ω_s = angular speed of shaft rotation

Superscripts

$(a),(b),(f),(I)$ = belonging to a particular region in the rotary machine
G = concerned with gyroscopic moment
M = associated with the mass points
R = associated with rotary machine
S = associated with the structure
$(\ )'$ = transpose of the quantity ( )

Presented as Paper 72-375 at the AIAA/ASME/SAE 13th Structures, Structural Dynamics, and Materials Conference, San Antonio, Texas, April 10–12, 1972; submitted April 20, 1972; revision received October 16, 1972. The author acknowledges the assistance of D. A. Wesley, Manager, Structural Mechanics Branch of Gulf General Atomic, in obtaining computer results for the example problem.

Index categories: Structural Dynamic Analysis; Aircraft Vibration.

*Group Leader, Structural Dynamics Group, Structural Mechanics Branch.

Introduction

THE increasing popularity of using large rotary machines for industrial applications has created many problems in structural dynamics. Some of them are closely associated with the phenomenon of dynamic interaction between an elastic structure and a deformable rotary machine. This structure-rotor interaction problem has attracted considerable attention from both the industrial and technical communities for the reason that the phenomenon tends to significantly amplify the dynamic loads experienced by the rotary machine and its local supporting members. The subject of airframe-engine interaction, for example, is being extensively studied by the aircraft manufacturers that must resolve the design problems of superjets powered by the giant-sized engines. Similar problems are also well-recognized in various other fields of industry where huge rotary machines are employed.[1]

The dynamic behavior of a flexible rotary machine interacting with other structural systems is a matter of great complexity. Some investigators recently approached this problem by highly idealizing the structure part of the coupled system. Leve and Biehl,[2] for instance, studied the airplane engine response with the airplane flexibility characterized by a massless cantilever beam. On the other

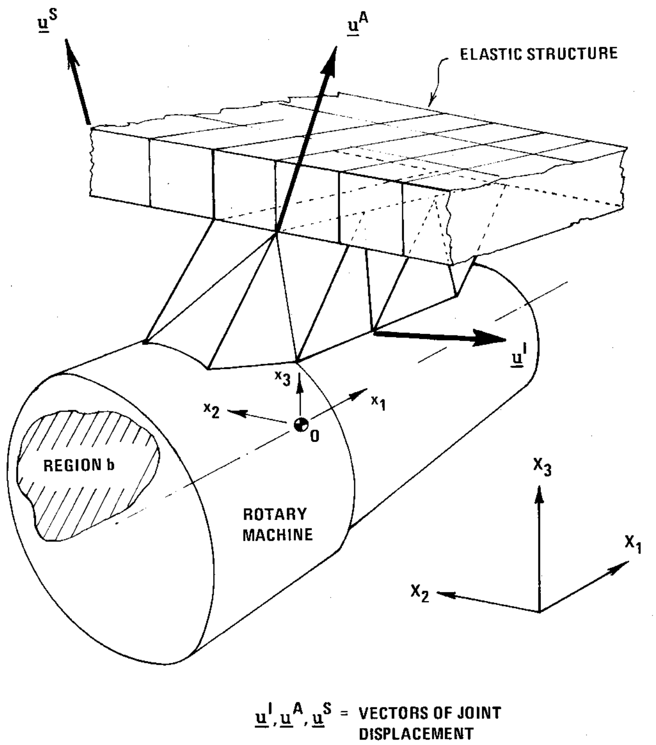

Fig. 1 Coordinate system and geometry.

hand, crudely modeled rotary machines were used in conjunction with elaborated structural representation. Presumably, a general solution which accounts for the details of both components may be obtained by using the current component mode synthesis techniques but the analytical procedures have presented difficulty for a number of reasons. In practice, the structure modes are frequently generated from a model which contains rotary machines represented as rigid mass lumps. In this case, the rigid-body displacements of the rotary machines are identified with the structural modes. Since the customary component mode synthesis methods[3-6] were developed for coupling distinct regions with no modal overlap among the components, the techniques are not directly applicable when the structure modes are supplied in this form. Additional work must be performed to make these modes rotorless prior to the synthesis of modes, and these extra steps were found to be time-consuming. (For simplicity, the terms rotary machine and rotor are used here interchangeably.) In the present paper, a method is presented which couples the free-free flexible rotor modes directly with the structure modes derived from the model containing the rigid masses of the rotors. The final system equations of motion involve several generalized matrices which are derived through modification of the energy quantities. The analysis begins with the development of transformation equations which relate the rotor displacements to the system modal coordinate vector. The use of free-free rotor modes decouples the system inertia matrix. The system stiffness matrix was derived by modifying the strain energy of the coupled system. The two groups of modes were found to be coupled in the final equations through the local stiffness matrices of the supporting structure. The matrix manipulation for this part requires greater effort which may be lightened with the aid of tensor notation. An interesting formulation is given for the derivation of system gyroscopic matrix based on a new moment-velocity relation.

Analysis

Let the analysis be confined, for the moment, to an interaction problem involving a single rotary machine, which is attached to an elastic structure as shown in Fig. 1. It is assumed that the structure is modeled by a lumped-parameter system, and the rotary machine may be represented by either a discrete model or a continuum. The derivation of the governing equations is carried out within the framework of small displacements.

Two Cartesian coordinate systems X_i and x_i ($i = 1,2,3$) are introduced, the former being the inertial reference frame and the latter a local reference frame considered to be fixed in the rotary machine. The local frame x_i, with origin designated as O, is assumed to coincide with the principal axes of the rotary machine in its undeformed state. This frame moves with the rigid-body motion of the rotor (relative to the X_i reference system). The elastic deformation of the rotary machine can then be conveniently characterized by introducing displacements relative to the x_i frame.

A column matrix u^R is introduced as the matrix whose elements correspond to the six rigid-body degrees of freedom of the rotor. The six degrees of freedom are the three translational displacement components of the point O and the three components of small rotation of the local frame x_i. Since the structure modes are assumed to be generated from a model containing the rigid rotary machine, the rigid-body displacements of the rotor may then be identified with the generalized modal coordinates of the structure component as

$$u^R = \phi^R q^S \tag{1}$$

where ϕ^R is the modal matrix of the structure for the rigid-body degrees of freedom of the rotary machine and q^S is the modal generalized coordinates.

Consider, for example, the case in which the rotary machine is also modeled by a discrete lumped-mass system with its modal matrix denoted by Φ, which defines the rotor displacements relative to the x_i reference frame. The absolute displacements of all the discrete points in the rotor may be represented by a column matrix which can be written as

$$\eta = Au^R + \Phi q^R \tag{2}$$

in which A is the rigid-body transformation matrix and q^R is the generalized modal coordinates associated with the rotor flexible modes.

We define q as the generalized modal coordinates of the interaction system, and the elements in q are arranged in the following order

$$q = \begin{Bmatrix} q^S \\ \hdashline q^R \end{Bmatrix} \tag{3}$$

Substituting Eq. (1) into Eq. (2) in conjunction with Eq. (3) gives

$$\eta = [A\phi^R \,\vdots\, \Phi]q \tag{4}$$

which relates the total displacements of the points in the rotor to the generalized coordinates q. Often the discrete points that one is dealing with belong to a particular set or region; therefore, Eq. (4) may appear in the form

$$\eta^{(b)} = [A^{(b)}\phi^R \,\vdots\, \Phi^{(b)}]q \tag{5}$$

where the superscript b indicates that the matrices are associated with the points in the region b and $\eta^{(b)}$ is a subset of η. Equations of this type relating a subset of η to the generalized coordinates of the system will be used often later on to derive the energy expressions of the interaction model. The final governing equations are formulated by substituting these energy expressions into the Lagrange's equations.

Inertia Matrix

The inertia matrix of the coupled system is derived from the system's kinetic energy T, which may be written as

$$T = 1/2\,\dot{\xi}'m^S\dot{\xi} + 1/2\,\dot{\eta}^{(m)\prime}m^R\dot{\eta}^{(m)} \tag{6}$$

in which ξ is the column matrix of displacements for the mass points in the system excluding the rotor mass and $\eta^{(m)}$ is a subset of η. The elements in $\eta^{(m)}$ are the displacements of those mass points in the rotor. The dot and prime denote, respectively, the time derivative and the transpose. It should be pointed out here that the additional inertia effects due to the rotation of the rotor shaft about its own axis are not included in Eq. (6) because these effects give rise to the gyroscopic coupling matrix, which will be dealt with in a later section of the paper.

Transforming ξ into the modal coordinates, one obtains

$$\dot{\xi} = [\phi^M \mid 0]\dot{q} \tag{7}$$

where ϕ^M is the structure modal matrix for the displacements of the mass points in the structure (not including rotor mass lump). The $\dot{\eta}^{(m)}$ is transformed in accordance with Eq. (5) as

$$\dot{\eta}^{(m)} = [A^{(m)}\phi^R \mid \Phi^{(m)}]\,\dot{q} \tag{8}$$

The modal transformation Eqs. (7) and (8) transform the kinetic energy Eq. (6) into

$$T = 1/2\,\dot{q}'\bar{M}\dot{q} \tag{9}$$

where

$$\bar{M} = \begin{bmatrix} \bar{m}^{SS} & \bar{m}^{SR} \\ \bar{m}^{RS} & \bar{m}^{RR} \end{bmatrix} \tag{10}$$

in which

$$\bar{m}^{SS} = \phi^{M\prime}m^S\phi^M + \phi^{R\prime}A^{(m)\prime}m^RA^{(m)}\phi^R \tag{11}$$

$$\bar{m}^{RR} = \Phi^{(m)\prime}m^R\Phi^{(m)} \tag{12}$$

$$\bar{m}^{SR} = \bar{m}^{RS\prime} = \phi^{R\prime}A^{(m)\prime}m^R\Phi^{(m)} \tag{13}$$

It can be readily verified that $\bar{m}^{SS}$ as defined by Eq. (11) is just the generalized mass matrix in q^S coordinate system for the structure component which contains the rigid rotor mass. Equation (12) is the expression for the generalized mass matrix in q^R coordinate system associated with the flexible rotor modes. In the present analysis, $\bar{m}^{SS}$ and $\bar{m}^{RR}$ are input matrices whose elements are to be generated from two separate eigenvalue-eigenvector computer programs. The terms $\bar{m}^{SR}$ and $\bar{m}^{RS}$ denote the inertia coupling matrices. If the rotor flexible modes are free-free modes, then the momentum relations that the elastic free-free modal displacements must satisfy can be shown to lead to the equation

$$A^{(m)\prime}m^R\Phi^{(m)} = 0 \tag{14}$$

This means that the submatrices $\bar{m}^{SR}$ and $\bar{m}^{RS}$ are null matrices, namely,

$$\bar{m}^{SR} = \bar{m}^{RS} = 0 \tag{15}$$

One then gets a simplified form for the generalized inertia matrix $\bar{M}$. If, in addition, the modes used are the natural modes of vibration, then $\bar{m}^{SS}$ and $\bar{m}^{RR}$ are both diagonal by virtue of the orthogonality of these modes. In the present analysis, we shall consider the case in which the elastic rotor modes are free-free and the orthogonality relation holds for the modes so that $\bar{M}$ is diagonal.

Stiffness Matrix

The strain energy of the entire system may be written as

$$U = 1/2\,u'ku + 1/2\,q^{R\prime}k^Rq^R \tag{16}$$

in which the first term represents the strain energy in the elastic structure and the second term is associated with the strain energy of elastic deformation of the rotor. The column matrix u contains elements corresponding to the displacements of joints in the elastic structure. The strain energy in the rotor has been expressed in terms of the generalized modal coordinates q^R. The terms k and k^R are the associated stiffness matrices in the u and q^R coordinates, respectively.

Let us assume that u has the partitioned form in accordance with the arrangement shown in Fig. 1, namely,

$$u = \begin{Bmatrix} u^S \\ u^A \\ u^I \end{Bmatrix} \tag{17}$$

Then the k matrix, which is partitioned consistent with Eq. (17), may be put in the form[7]

$$k = \begin{bmatrix} k^{SS} & k^{SA} & \\ k^{AS} & k^{AA} & k^{AI} \\ & k^{IA} & k^{II} \end{bmatrix} \tag{18}$$

The modal transformation equations for u^S and u^A are

$$u^S = \phi^Sq^S \qquad u^A = \phi^Aq^S \tag{19}$$

where ϕ^S and ϕ^A are the corresponding modal matrices. The u^I is assumed to be compatible with the displacements of the rotor at the interface, and its modal transformation, in accordance with Eq. (5), is therefore made with the equation

$$u^I = \eta^{(I)} = [A^{(I)}\phi^R \mid \Phi^{(I)}]\,q \tag{20}$$

Combining Eq. (20) with Eq. (19), one can relate u to the generalized modal coordinate system q by the transformation

$$u = T^1q \tag{21}$$

where

$$T^1 = \begin{bmatrix} \phi^S & 0 \\ \phi^A & 0 \\ A^{(I)}\phi^R & \Phi^{(I)} \end{bmatrix} \tag{22}$$

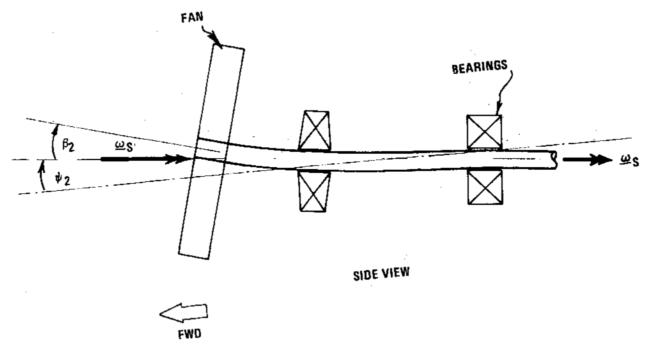

Fig. 2 Idealized motion of the fan-shaft system.

When Eq. (21) is substituted into Eq. (16), the strain energy of the coupled system becomes

$$U = 1/2\, q' T^{1\prime} k T^1 q + 1/2\, q^{R\prime} k^R q^R$$
$$= 1/2\, q' \bar{K} q \qquad (23)$$

where $\bar{K}$ is the generalized stiffness matrix of the coupled system and its partitioned form is consistent with Eq. (10), i.e.,

$$\bar{K} = \left[\begin{array}{c|c} \bar{k}^{SS} & \bar{k}^{SR} \\ \hline \bar{k}^{RS} & \bar{k}^{RR} \end{array}\right] \qquad (24)$$

In Eq. (24) the submatrices are defined as

$$\bar{k}^{SS} = T^{2\prime} k T^2 \qquad (25)$$

with

$$T^2 = \left[\begin{array}{c} \phi^S \\ \hline \phi^A \\ \hline A^{(I)} \phi^R \end{array}\right] \qquad (26)$$

$$\bar{k}^{RR} = \Phi^{(I)\prime} k^{II} \Phi^{(I)} + k^R \qquad (27)$$

$$\bar{k}^{SR} = \bar{k}^{RS\prime}$$
$$= \phi^{A\prime} k^{AI} \Phi^{(I)} + \phi^{R\prime} A^{(I)\prime} k^{II} \Phi^{(I)} \qquad (28)$$

It can be shown that $\bar{k}^{SS}$ as defined by Eqs. (25) and (26) is simply the generalized stiffness matrix in the q^S coordinate system for the structure model which contains the rotary machine as rigid mass lump. If natural modes of vibration are used, then $\bar{k}^{SS}$ and k^R are diagonal. It is understood that these two diagonal matrices are related, respectively, to the two diagonal mass matrices $\bar{m}^{SS}$ and $\bar{m}^{RR}$ through the natural frequencies. The ith element in $\bar{k}^{SS}$ and the rth element in k^R are

$$\bar{k}^{SS} = \omega_i^2 \bar{m}_i^{SS}$$
$$k_r^R = \Omega_r^2 \bar{m}_r^{RR} \qquad (29)$$

Gyroscopic Matrix

The gyroscopic matrix accounts for the gyroscopic coupling effects resulting from the motion of the fan-shaft system, which is spinning with an angular velocity ω_S. For simplicity, let us consider the gyroscopic effects contributed from the motion of the fan only and assume that the idealized geometry of the fan-shaft system is as shown in Fig. 2. The relation between the components of gyroscopic moment and the components of rotational velocity of the fan station is established first. In the conventional treatment, the direction of the spin vector ω_S has often been taken to be perpendicular to the fan plane. In the present analysis, the actual situation is more closely approximated by assuming that the direction of the spin vector moves with the rigid-body reference position of the shaft. This means that the ω_S vector is parallel to the x_1 axis as shown in Fig. 2. Following the scheme in Ref. 8, the following moment-velocity relation can be deduced

$$\begin{Bmatrix} f_2^G \\ f_3^G \end{Bmatrix} = \omega_s \begin{bmatrix} 0 & I_1^F \\ -I_1^F & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi}_2 \\ \dot{\psi}_3 \end{Bmatrix}$$
$$+ \omega_s \begin{bmatrix} 0 & I_1^F - I_2^F \\ -(I_1^F - I_2^F) & 0 \end{bmatrix} \begin{Bmatrix} \dot{\beta}_2 \\ \dot{\beta}_3 \end{Bmatrix} \qquad (30)$$

For compactness, Eq. (30) may be written symbolically as

$$f^G = \omega_s (J^S \dot{\psi} + J^R \dot{\beta}) \qquad (31)$$

where J^S and J^R are the two fan inertia matrices in the first and second terms of Eq. (30) and the symbols f^G, $\dot{\psi}$ and $\dot{\beta}$ stand for the three column matrices in that equation. The column matrix ψ, whose elements are the two components of rotation associated with the rigid-body motion of the rotor, may be extracted from u^R in the following manner

$$\dot{\psi} = T^3 \dot{u}^R \qquad (32)$$

with

$$T^3 = [0 \mid I_{2\times 2}]$$

Substituting Eq. (32) into Eq. (31) and making coordinate transformation yields

$$f^G = \omega_s [J^S T^3 \phi^R \mid J^R B^{(f)}] q \qquad (33)$$

which relates the gyroscopic moment vector to the q coordinate system. The new matrix $B^{(f)}$ introduced here is the matrix for the rotor modal rotation at the fan station.

The gyroscopic matrix of the interaction system may be derived from the virtual work considerations by using Eq. (33). The virtual work done by the gyroscopic moments, δW^G, is given by

$$\delta W^G = f^{G\prime} \delta\gamma = Q^{G\prime} \delta q \qquad (34)$$

where $\delta\gamma$ is the virtual rotation at the fan station and Q^G is the generalized force column matrix associated with the gyroscopic moments. The virtual rotation at the fan station is simply

$$\delta\gamma = \delta\psi + \delta\beta$$
$$= [T^3 \phi^R \mid B^{(f)}] \delta q \qquad (35)$$

Substituting Eq. (35) into Eq. (34) leads to the following equation for the generalized force vector Q^G

$$Q^G = \left[\begin{array}{c} \phi^{R\prime}\, T^{3\prime} \\ \hline B^{(f)\prime} \end{array}\right] f^G \qquad (36)$$

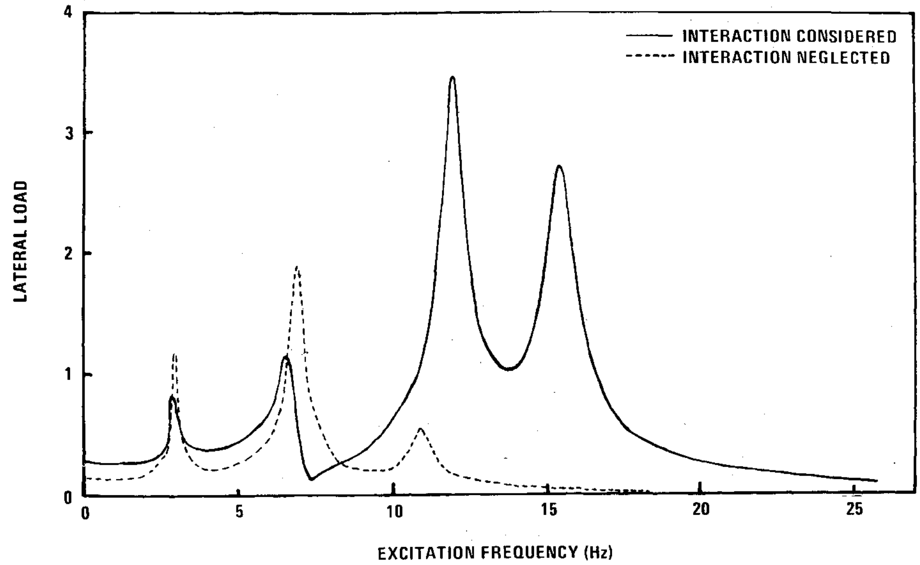

Fig. 3 Interaction effects on dynamic load (lateral component).

Replacing f^G in the preceding equation by Eq. (33), one arrives at

$$Q^G = \omega_s \bar{G} \dot{q} \tag{37}$$

where $\bar{G}$ is the generalized gyroscopic matrix which may be put in the partitioned form as

$$\bar{G} = \left[\begin{array}{c|c} \bar{G}^{SS} & \bar{G}^{SR} \\ \hline \bar{G}^{RS} & \bar{G}^{RR} \end{array}\right] \tag{38}$$

in which the submatrices are

$$\bar{G}^{SS} = \phi^{R\prime} T^{3\prime} J^S T^3 \phi^R \tag{39}$$

$$\bar{G}^{SR} = \phi^{R\prime} T^{3\prime} J^R B^{(f)} \tag{40}$$

$$\bar{G}^{RS} = B^{(f)\prime} J^S T^3 \phi^R \tag{41}$$

$$\bar{G}^{RR} = B^{(f)\prime} J^R B^{(f)} \tag{42}$$

It is interesting to note that the assumption introduced in the present analysis regarding the direction of spin vector leads to a generalized gyroscopic matrix $\bar{G}$ in which

$$\bar{G}^{SR} \neq \bar{G}^{RS\prime} \tag{43}$$

Time-Dependent Generalized Forces

The virtual work method demonstrated in the previous section can be used similarly to derive the column matrix for the generalized force associated with the applied time-dependent forces. Consider, for example, the virtual work done by a set of time-dependent point forces, $\{f(t)\}$, applied at those points in some specified region a in the rotary machine. The virtual displacements in the directions corresponding to those of $\{f(t)\}$ are collected to form a column matrix defined as

$$\delta\eta^{(a)} = [A^{(a)} \phi^R \,|\, \Phi^{(a)}] \delta q \tag{44}$$

which is written in accordance with Eq. (5). The virtual work expression is given by

$$\begin{aligned} \delta W &= \{f(t)\}' \, \delta\eta^{(a)} \\ &= Q(t)' \, \delta q \end{aligned} \tag{45}$$

where $Q(t)$ is the generalized force vector corresponding to the applied forces. Making use of Eq. (44), it can be readily shown that

$$Q(t) = \left[\begin{array}{c} \phi^{R\prime} A^{(a)\prime} \\ \hline \Phi^{(a)\prime} \end{array}\right] f(t) \tag{46}$$

System Equations of Motion

Substitution of all the energy quantities and virtual work expressions derived into the Lagrange's equations yields the equations of motion for the coupled structure-

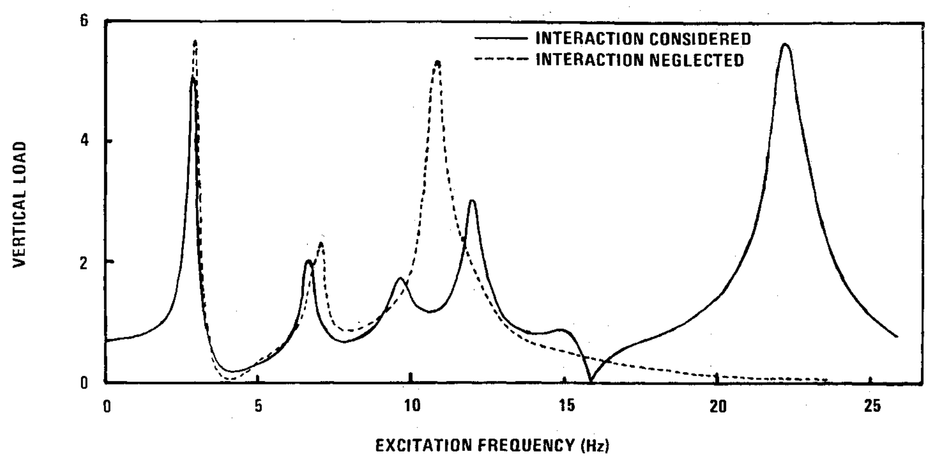

Fig. 4 Interaction effects on dynamic load (vertical component).

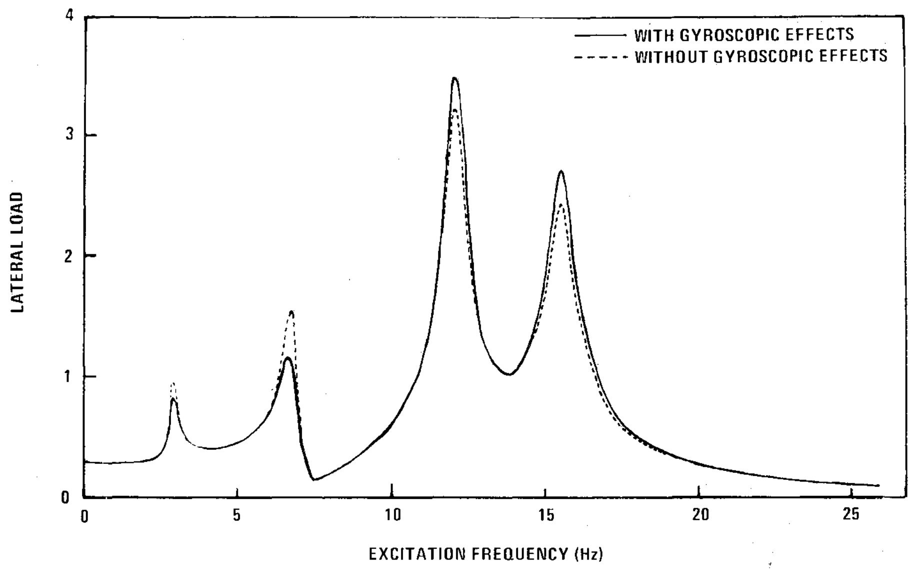

Fig. 5 Influence of gyroscopic coupling matrix on dynamic load (lateral component).

rotor system as

$$\bar{M}\ddot{q} + \bar{K}q - \omega_s \bar{G}\dot{q} = Q(t) \qquad (47)$$

which can be solved by established methods. The method of complex frequency response matrix was used to obtain the steady-state solution to Eq. (47) for the example problem.

Numerical Example

An example problem is solved to illustrate the validity of the analysis. Fictitious airplane modes are used to interact with free-free modes of a pseudo medium-sized aircraft engine. To obtain informative results, the natural frequencies of modes from the two groups are assumed to be distributed in such a way that, before they are coupled, the highest airplane structure mode has its frequency near the fundamental frequency of the engine free-free modes. For this particular example, each group is assumed to have three modes. The frequencies of the airplane structure are chosen to be 3, 7, and 11 Hz, and those of the rotary machine modes are taken to be 10, 14, and 17 Hz. The analysis was performed for harmonic response excited by two unit harmonic forces assumed to be acting at the fan station of the engine. The excitation simulates the disturbances from fan unbalance. The direction of the unit force vector is such that one of its components is acting in the lateral direction (along the x_2 axis) and the other in the vertical direction (x_3 axis). The dynamic amplification is determined as described above for the engine supporting member loads, and the computations are made with the computer code STROM, which stands for the interaction between structure and rotary machine. The speed of the fan rotating about the shaft axis is assumed to be 4500 rpm.

Figures 3 and 4 display the steady-state results computed for the load amplification factor at some joint near the forward mounting point of the engine. Curves are plotted for loads in both lateral and vertical directions. In each figure, the curve generated from the interaction analysis is compared with the case without interaction effects. In the latter case, the coupling with the rotor modes is removed and only a rigid engine is attached to the airframe. As revealed by these diagrams, the frequency-response characteristics of the structural member loads can be significantly altered by the dynamic interaction effects.

The gyroscopic effects on the dynamic responses were investigated and the results are given in Figs. 5 and 6 for the same problem. The influence of gyroscopic coupling matrix tends to increase the response at peaks associated with the rotor modes and lower the response at peaks associated with the structure modes. The changes are seen to be smooth and gradual. This behavior is in contrast to that previously presented in Ref. 2, where the gyroscopic effects exhibited considerable irregularity.

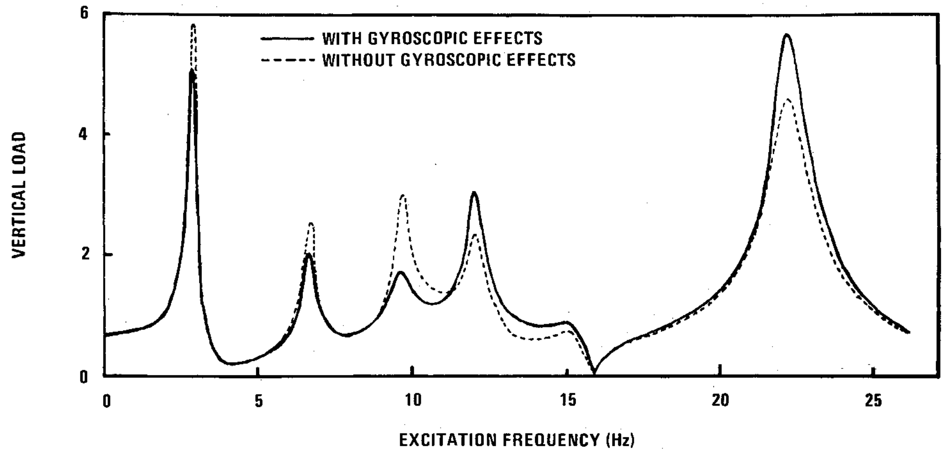

Fig. 6 Influence of gyroscopic coupling matrix on dynamic load (vertical component).

Concluding Remarks

A theoretical solution to the structure-rotor interaction problem has been obtained through synthesis of component modes. The fact that the modal coupling can be made directly without first removing the rigid rotor masses from the structure modes enables the analysts to generate the required information at reduced time and cost. Matrices corresponding to aerodynamic effects may be appended to the system equations of motion if the inclusion of such effects is deemed desirable. The derivation of these matrices, however, is not reiterated here because of space limitation. The extension to cases involving multiple rotary machines can be done by expanding the dimension of q vector. Such modification, although requiring some labor, is relatively straightforward.

References

[1]Rizk, W. and Seymour, D. F., "Investigations into the Failure of Gas Circulators and Circuit Components at Hinkley Point Nuclear Power Station," *Proceedings, Institutions of Mechanical Engineers,* Vol. 179, Pt. 1, No. 21, 1964–65, pp. 627–673.

[2]Leve, H. L., and Biehl, F. A., "Aircraft Engine Response Due to Fan Unbalance," Douglas Paper 5660, Oct. 6–10, 1969, McDonnell Douglas Corp., Long Beach, Calif.

[3]Hurty, W. C., "Dynamic Analysis of Structural Systems Using Component Modes," *AIAA Journal,* Vol. 3, No. 4, April 1965, pp. 678–685.

[4]Craig, R. R., and Bampton, M. C. C., "Coupling of Substructures for Dynamic Analysis," *AIAA Journal,* Vol. 6, No. 7, July 1968, pp. 1313–1319.

[5]Goldman, R. L., "Vibration Analysis by Dynamic Partitioning," *AIAA Journal,* Vol. 7, No. 6, June 1969, pp. 1152–1154.

[6]Benfield, W. A., and Hruda, R. F., "Vibration Analysis of Structures by Component Mode Substitution," *AIAA Journal,* Vol. 9, No. 7, July 1971, pp. 1256–1261.

[7]Przemieniecki, J. S., "Matrix Structural Analysis of Substructures," *AIAA Journal,* Vol. 1, No. 1, Jan. 1963, pp. 138–147.

[8]Tong, K. N., *Theory of Mechanical Vibration,* Wiley, New York, 1961, pp. 150–151.